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A transient flow problem in magnetohydrodynamics

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A flat plate $x = 0$, $|y| < L$ is initially at rest in an electrically conducting, inviscid, incompressible fluid permeated by a uniform magnetic field $(B_0, 0, 0)$. The plate is impulsively accelerated to a small velocity $(-U, 0, 0)$ which is then kept constant. It is assumed that $LV/\lambda \gg 1$ and $U/V \ll 1$, where V is the Alfven velocity, and λ is the magnetic diffusivity.

Four stages in the development of the flow are distinguished, the last three being:

(ii) $L \gg Vt \gg (\lambda t)^{\frac{1}{2}}$. During this stage the initial potential flow is being disturbed by propagation of electric current and vorticity from the plate. The initial discontinuity on the plate has only propagated a small distance away compared to L, but a large distance compared to the length scale of diffusion $(\lambda t)^{\frac{1}{2}}$. Exact solutions to the flow are found in the neighbourhood of $y = -L$ and $y = 0$ $(x = 0)$.

(iii) $Vt \gg L \gg (\lambda t)^{\frac{1}{2}}$. The asymptotic behaviour of the electric current and vorticity on the plate are determined showing that a column of fluid of length Vt moves with the plate, aligned to the magnetic field. The transverse diffusion of the current sheets bounding the column accelerates the fluid in layers of thickness $O(\lambda t)^{\frac{1}{2}}$.

(iv) $(\lambda t)^{\frac{1}{2}} \gg L$. Unlike stages (ii) and (iii), where the motion is dominated by the propagation of vorticity and electric current as Alfven waves from the plate, diffusive mechanisms completely dominate the motion. 'Slug' flow is maintained. The nature of the flow, including the structure of the layers bounding the column are determined for $|\mathbf{x}| \ll (\lambda t)^{\frac{1}{2}}$.

1. Introduction

When a rigid body is impulsively jerked into motion in an incompressible non-conducting fluid that is initially at rest, an irrotational flow is instantaneously established by the impulsive pressure distribution. Vorticity then diffuses (through viscous forces) into the fluid from the surface of the body, and is convected by the irrotational stream past the body, and ultimately by its own self-induced velocity field.

In a conducting fluid permeated by a uniform magnetic field, there is a further mechanism by which the flow may acquire vorticity, and which may (if the field is strong enough) totally dominate the development of the flow. This mechanism is the propagation of vorticity (and electric current) by Alfven waves. Immediately after the impulsive start to the motion, there is a vortex sheet on the body. This discontinuity may subsequently be unacceptable to the body and propagate away as a wave in both directions along the applied field, with the Alfven velocity. Diffusive mechanisms (viscosity, and now also finite conductivity) will modify the vorticity distribution, as will convection of vorticity by the total velocity field. But it might be anticipated that if the field is very strong, the Alfven wave mechanism will be the dominant process by which vorticity can penetrate the fluid, at least for some considerable time.

Certain aspects of the problem have been studied by previous authors. Stewartson (1956) considered the flow set up in an inviscid fluid by accelerating a perfectly conducting sphere from rest to a constant velocity parallel to the applied magnetic field. The governing equations were linearized on the assumption that this velocity was small compared to the Alfven velocity. Assuming there were no instabilities, a model for the asymptotic state was obtained in which a column of fluid (parallel to the magnetic field) moves with the sphere, while the velocity in the layer bounding the column continues to increase indefinitely. Ludford & Singh (1963) re-examined Stewartson's work and showed it to be incorrect, owing to the application of incorrect boundary conditions. However, by adopting a slightly different approach, an asymptotic solution was obtained which showed similar features, except that the velocity profile for the flow outside the column was less singular than Stewartson's. No determination was made of the acceleration of the fluid in the layer bounding the column but the order of magnitude obtained by Stewartson appears to be still qualitatively correct. In both papers, when obtaining the asymptotic behaviour, the electrostatic approximation was made implicitly (neglect of $\partial\chi/\partial t$ in equation (1.10) below). It follows that the effects of Alfven wave propagation, which are important for a finite time, do not appear in the asymptotic analysis.

Ludford & Leibovich (1965) considered the two-dimensional flow due to the impulsive motion of a non-conducting, thin airfoil aligned both to the flow and the applied magnetic field. The fluid was supposed perfectly conducting and inviscid. The governing equations were linearized (as in Stewartson 1956) on the assumption that the velocity of the airfoil was small compared to the Alfven velocity (in later papers (Leibovich & Ludford 1965, 1966), this approximation is not made and the non-linear terms are retained). The fluid velocity and perturbation magnetic field were separated into a potential part, resulting directly from the pressure distribution, and a wave-like part (independent of the pressure) corresponding to the emission of Alfven waves from the body both upstream and downstream (Stewartson 1960). The development of the flow was analyzed by noting that (unlike the potential disturbance) the Alfven waves may have a transverse length scale much shorter than the longitudinal length scale, as a result of the thinness of the airfoil. Asymptotically, a column of fluid was shown to move with the airfoil bounded by electric current and vortex sheets.

Since the main concern of the above authors has been with the ultimate flow, the work of Chester (1961), Chester & Moore (1961), and Glauert (1964) is also relevant. They were all concerned with the steady flow obtained when a body moves with uniform velocity through a viscous fluid aligned to the applied magnetic field. Chester & Moore showed that, for large Hartmann number a column of fluid moves with the body (as was anticipated by the above transient flow models), while outside the column the fluid is at rest. More explicit results were obtained by considering a disk placed perpendicular to the magnetic field, paying particular attention to the layer dividing the column from the fluid at rest. Glauert reconsidered this layer in greater detail. Instead of just assuming that the magnetic field was weakly perturbed, the full equations were retained and the usual boundary-layer approximations made. Assuming the magnetic Reynolds number was small, a first approximation was obtained in full agreement with the results of Chester & Moore. However, a second approximation could not be found satisfying the boundary conditions.

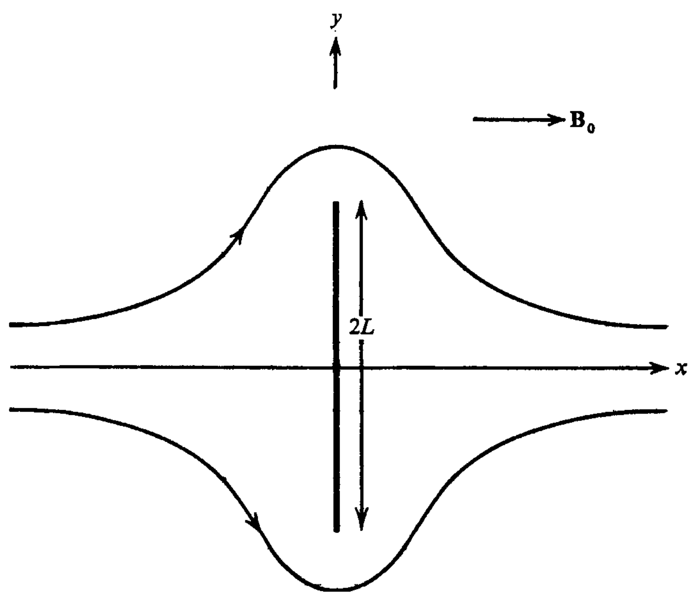

FIGURE 1. Initial streamlines for the prototype problem.

The aim of this paper is to provide an exact solution of a prototype problem which will exhibit (*a*) the acquisition of vorticity by the fluid through the Alfven wave mechanism (§§ 3 and 4), (*b*) the ultimate settling down to a steady state through the diffusive (Stewartson) mechanism (§§ 5 and 6). In order to isolate these mechanisms, both viscosity and non-linear effects will be neglected; this puts certain restrictions on the dimensionless numbers that characterize the flow (see equations (1.2), (1.3), (1.4)). The subsequent analysis extends the Ludford & Leibovich (1965) model of Alfven wave propagation from slender bodies to 'broad' bodies (§§ 3 and 4). The discussion of diffusive effects in this model demonstrates the 'early' stage of the diffusive (Stewartson) mechanism (§ 5). Thus the analysis provides a bridge between the previously unrelated work of Ludford & Leibovich (1965) and Stewartson (1956). The present choice of prototype problem enables the final (Stewartson) mechanism to be considered in greater detail than before (§ 6); particular attention is paid to the layer dividing the column of fluid from the flow in the outer regions.

Figure 1 indicates the configuration to be considered. Suppose that a flat plate of finite width $2L$ is immersed in an inviscid, incompressible fluid of infinite extent, with density ρ, electrical conductivity σ, and magnetic permeability μ, and suppose that the fluid is permeated by a uniform magnetic field $\mathbf{B}_0$ perpendicular to the plate. For time $t' < 0$, the plate and the fluid are at rest, and at the instant $t' = 0$, the plate is jerked into motion with uniform velocity $-\mathbf{U}$ parallel to $\mathbf{B}_0$. Relative to axes $Ox'y'$ moving with the plate (as in figure 1), the flow $\mathbf{u}'$ at time $t' = 0+$ is irrotational and $\mathbf{u}' \sim \mathbf{U}$ at a large distance. The streamlines are as indicated in the figure; there is a vortex sheet on each side of the plate. The problem is to determine the nature of the flow for all $t' > 0$.

Defining the Alfven velocity

$$\mathbf{V} = \mathbf{B}_0/\surd(\mu\rho), \tag{1.1}$$

the character of the flow depends on the magnitude of the two dimensionless numbers

$$A = U/V \quad \text{and} \quad \beta = \lambda/LV. \tag{1.2}$$

A is the Alfven number and β^{-1} is the Lundquist number. Moreover, the neglect of the viscous term $\nu\nabla^2\mathbf{u}$ in the equation of motion is justified provided the Hartmann number

$$M = VL/\surd(\nu\lambda) \gg 1. \tag{1.3}$$

It will be assumed throughout this paper that

$$A \ll 1 \quad \text{and} \quad \beta \ll 1. \tag{1.4}$$

These conditions are not totally unrealistic by laboratory standards, although a very strong magnetic field would be required to satisfy $\beta \ll 1$, in say, mercury with L of the order of centimetres. The condition $A \ll 1$ permits the total neglect (in a first approximation) of all the non-linear terms appearing in the governing equations.† This is not strictly justifiable in the neighbourhood of the edges $x' = 0, y' = \pm L$ of the plate, where $\mathbf{u}'$ is initially singular; non-linear effects must be important near these edges. If, however, a self-consistent description of the overall development of the flow is obtained on the basis of the linearized equations, it may reasonably be supposed that the non-linear effects may be treated as a localized perturbation, and that the description obtained is at any rate qualitatively correct. The difficulty encountered due to the singular behaviour near the ends of the plate would not appear in the case of say a circular cylinder. However, if progress is to be made analytically some simplification of the geometry is required, e.g. the slender body approach of Ludford & Leibovich (1965). Since an important motivation for this present study is the determination of the nature of Alfven wave propagation from a 'broad' body, a natural choice is a slender body placed perpendicular to the flow. Thus the consideration of a flat plate leads to two simplifications. First, the magnetic boundary conditions on the body, which are normally very involved, are simple. Secondly, the problem may be reduced to solving an integral equation (2.28).

† The approximation is similar to the acoustic approximation in gas dynamics; if a body is jerked into motion in a compressible (non-conducting) fluid, the equations may be linearized if the velocity is everywhere small compared to the speed of sound.

Let $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}'$ represent the perturbed magnetic field ($\mathbf{b}' = 0$ at $t' = 0$). We define dimensionless variables as follows:

$$\mathbf{x} = \mathbf{x}'/L, \quad t = (V/L)t', \tag{1.5}$$

$$\mathbf{u} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right) = (\mathbf{u}' - \mathbf{U})/U, \tag{1.6}$$

$$\mathbf{b} = \left(\frac{\partial\chi}{\partial y}, -\frac{\partial\chi}{\partial x}, 0\right) = \mathbf{b}'/(\sqrt{(\rho\mu)}\,U), \tag{1.7}$$

where ψ is the stream function and χ is the magnetic vector potential. The dimensionless vorticity and electric current distributions are

$$\boldsymbol{\omega} = (0, 0, -\nabla^2\psi), \quad \mathbf{j} = (0, 0, -\nabla^2\chi). \tag{1.8}$$

Neglecting the non-linear terms ($A \ll 1$) the equations governing the motion reduce to the well known form (cf. Stewartson 1956, equations (2.11) to (2.15))

$$\frac{\partial\omega}{\partial t} = \frac{\partial j}{\partial x}, \tag{1.9}$$

$$\frac{\partial\chi}{\partial t} = \frac{\partial\psi}{\partial x} + \beta\nabla^2\chi. \tag{1.10}$$

Equation (1.9) is the vorticity equation, the term on the right representing generation of vorticity by the rotational Lorentz force, and (1.10) may be recognized as Ohm's law with the electric current $E_z \propto \partial\chi/\partial t$.

If $\beta = 0$, (1.9) and (1.10) are simply equations describing Alfven wave propagation in an ideal fluid. Since disturbances in ω and j originate from the plate, considerations of symmetry indicate that the required solution is of the form

$$\left.\begin{aligned} \omega &= -j_0(y, t-x), \quad j = j_0(y, t-x) \quad (x > 0),\\ \omega &= -j_0(y, t+x), \quad j = -j_0(y, t+x) \quad (x < 0), \end{aligned}\right\} \tag{1.11}$$

where $j_0(y, \tau) = 0$ if $\tau < 0$. The determination of the function j_0 requires a consideration of the initial conditions, the boundary conditions and equation (1.10).

At the instant $t = 0+$, the flow is the unique irrotational flow past the plate and its stream function is given by

$$\psi(x, y, 0) = \psi_0(x, y) = \operatorname{Im}\{(1+z^2)^{\frac{1}{2}} - z\}, \tag{1.12}$$

where $z = x + iy$, while the perturbation magnetic field vanishes

$$\chi(x, y, 0) = 0. \tag{1.13}$$

Subsequently the flow is symmetrical fore and aft of the plate† and χ is skew-symmetric in x:

$$\left.\begin{aligned} \psi(-x, y) &= \psi(x, y),\\ \chi(-x, y) &= -\chi(x, y). \end{aligned}\right\} \tag{1.14}$$

On the plate

$$\psi + y = 0, \tag{1.15}$$

† This of course is no longer true if non-linear effects are included.

and the normal magnetic field perturbation $\partial\chi/\partial y$ is continuous across the plate; this condition together with (1.14) implies that

$$\chi = 0 \quad \text{on the plate,} \tag{1.16}$$

(indeed $\chi(0, y, t) = 0$ for all y, t). Further the symmetry condition implies that

$$j = 0 \quad \text{for} \quad x = 0, \quad |y| > 1. \tag{1.17}$$

Finally,

$$\psi \to 0, \quad \chi \to 0 \quad \text{as} \quad |z| \to \infty. \tag{1.18}$$

The equations (1.8) to (1.10), the initial conditions (1.12), (1.13) and the boundary conditions (1.15) to (1.18) complete the mathematical statement of the problem.

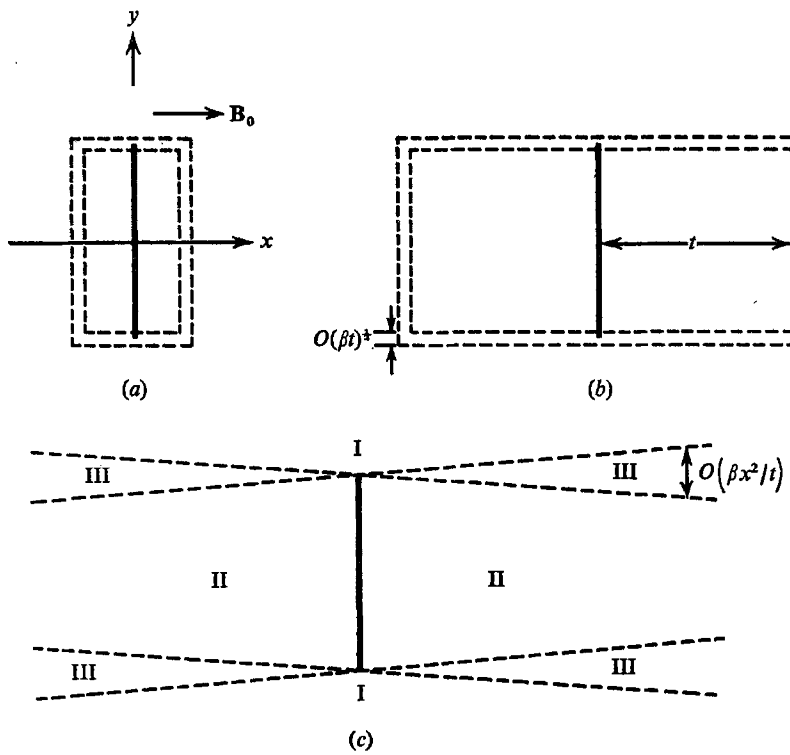

FIGURE 2. Pictorial representation of the stages (ii), (iii) and (iv) in the development of the flow. (a) Stage (ii), $\beta \ll t \ll 1$. (b) Stage (iii), $1 \ll t \ll \beta^{-1}$. (c) Stage (iv), $t \gg \beta^{-1}$.

The subsequent analysis reveals that four stages can be distinguished in the development of the flow, and it may be as well to provide, in anticipation, a summary of the broad physical characteristics of the development during these stages. Remembering that $\beta \ll 1$, the stages of development are as follows:

(i) $t \ll \beta$. At this early stage of the flow, the initial diffusing current sheet of thickness $O(\beta t)^{\frac{1}{2}}$ is not yet clear of the plate, since $t \ll (\beta t)^{\frac{1}{2}}$. At this stage there is some similarity with the Rayleigh problem, as treated by Dix (1963). The details are not of very great interest in the present situation and are not analyzed in this paper.

(ii) $\beta \ll t \ll 1$. During this stage, the initial Alfven wave is well clear of the plate, but its distance from the plate is still small compared with 1 (figure 2(*a*)). The influence of the finite width of the plate is not yet felt as far as the flow (*a*) in the neighbourhood of the edges $y = \pm 1$ and (*b*) in the neighbourhood of the centre $y = 0$ of the plate is concerned. The exact solution for the development of the flow during this stage in these two regions is given in § 3.

(iii) $1 \ll t \ll \beta^{-1}$. During this stage the initial Alfven waves centred on $x = \pm t$, $|y| < 1$ are far from the plate. The diffusion length scale $(\beta t)^{\frac{1}{2}}$ is still, however, small compared with the width of the plate, and the electric current, travelling as an Alfven wave, particularly in the neighbourhood of the planes $y = \pm 1$ has not yet diffused much in the lateral directions (figure 2(*b*)). This stage of the development is analyzed in §§ 4 and 5.

(iv) $\beta^{-1} \ll t$. By this final stage (see § 6), diffusion effects have had time to establish an asymptotic state, at any rate for $|\mathbf{x}| \ll (\beta t)^{\frac{1}{2}}$. The Alfven wave mechanism propagates electric current a distance $O(t)$ in the x direction. However its influence on the fluid flow is negligible, since its strength is small $O(\beta t)^{-2}$ due to transverse diffusion. The asymptotic state for $|\mathbf{x}| \ll (\beta t)^{\frac{1}{2}}$ is steady except for small regions within a distance $O(\beta x^2/t)^{\frac{1}{2}}$ of the planes $y = \pm 1$ (figure 2(*c*)); within these regions the fluid continues to accelerate, the velocity being $O(t/\beta x^2)^{\frac{1}{4}}$. (Viscous effects would modify this result.)

2. The integral equation

With the aid of transform techniques the problem stated in § 1 may be reduced to that of determining the solution of an integral equation (2.28). In this section this integral equation is derived together with other basic equations required in the subsequent sections.

We define Fourier and Laplace transforms

$$\tilde{\phi}(k) = \int_{-\infty}^{\infty} \phi(y)\, e^{iky}\, dy, \tag{2.1}$$

$$\hat{\phi}(p) = \int_{0}^{\infty} \phi(t)\, e^{-pt}\, dt \quad (\mathrm{Re}\, p > 0), \tag{2.2}$$

and for clarity express the double transform as $\bar{\phi}(k, p)$. The inverses of the transforms are

$$\phi(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}(k)\, e^{-iky}\, dk, \tag{2.3}$$

$$\phi(t) = \frac{1}{2\pi i}\int_{C} \hat{\phi}(p)\, e^{pt}\, dp, \tag{2.4}$$

where the contour C is taken from $-i\infty$ to $i\infty$, to the right of any cuts and poles of $\hat{\phi}(p)$.

Taking the double transforms of the governing equations (1.8) to (1.10) leads to

$$p\bar{\omega} - \frac{\partial}{\partial x}\bar{\jmath} = 0, \tag{2.5}$$

$$p\overline{\chi}-\frac{\partial}{\partial x}\overline{\psi}=\beta\left(\frac{\partial^2}{\partial x^2}-k^2\right)\overline{\chi}, \tag{2.6}$$

$$\left(\frac{\partial^2}{\partial x^2}-k^2\right)(\overline{\psi},\overline{\chi})=-(\overline{\omega},\overline{j}), \tag{2.7}$$

while the boundary conditions become

$$\left.\begin{array}{c}\overline{\chi}=0,\\ \dfrac{\partial\widehat{\psi}}{\partial y}=\dfrac{-1}{p}\,(|y|<1),\quad \hat{j}=0\quad(|y|>1)\end{array}\right\}\quad\text{on}\quad x=0, \tag{2.8}$$

$$\overline{\chi},\overline{\psi},\text{etc.}\to 0\quad\text{as}\quad |x|\to\infty. \tag{2.9}$$

The region $x>0$ is considered. Since equations (2.5) to (2.7) are fourth-order ordinary differential equations for $\overline{\psi}$, $\overline{\chi}$ in terms of x, they have particular integrals e^{nx}, where n takes the values

$$n=\pm|k|,\quad \pm\lambda, \tag{2.10}$$

and where

$$\lambda=\left\{\frac{p^2+\beta pk^2}{1+\beta p}\right\}^{\frac{1}{2}}. \tag{2.11}$$

Moreover, λ has the important property

$$\operatorname{Re}\lambda(k,p)>0\quad\text{when}\quad k\text{ is real and }\operatorname{Re}p>0. \tag{2.12}$$

The two positive values are immediately excluded from the solution by the conditions as $x\to\infty$. Thus $\overline{\psi}$ and $\overline{\chi}$ are given by

$$\overline{\psi}=A_1(k,p)\,e^{-\lambda x}+A_2(k,p)\,e^{-|k|x}, \tag{2.13}$$

$$\overline{\chi}=B_1(k,p)\,e^{-\lambda x}+B_2(k,p)\,e^{-|k|x}. \tag{2.14}$$

The ratio of the constants is determined by equations (2.5) to (2.7) together with the boundary condition $\overline{\chi}=0$ on $x=0$. Hence in terms of $j_0(y,t)$ the electric current on $x=0+$, ω, j, etc., are given by

$$\overline{\omega}(x,k,p)=-(\lambda/p)\,\overline{j}_0(k,p)\,e^{-\lambda|x|}, \tag{2.15}$$

$$\overline{j}(x,k,p)=(\operatorname{sgn}x)\,\overline{j}_0(k,p)\,e^{-\lambda|x|}, \tag{2.16}$$

$$\overline{\psi}(x,k,p)=-(1+\beta p)\frac{\overline{j}_0(k,p)}{k^2-p^2}\left\{\frac{\lambda}{p}\,e^{-\lambda|x|}-\frac{p}{|k|}\,e^{-|k||x|}\right\}, \tag{2.17}$$

$$\overline{\chi}(x,k,p)=(\operatorname{sgn}x)\,(1+\beta p)\frac{\overline{j}_0(k,p)}{k^2-p^2}\{e^{-\lambda|x|}-e^{-|k||x|}\}. \tag{2.18}$$

The Fourier transforms (2.15) to (2.18) are made analytic in a strip containing the real axis by defining $|k|$ as

$$|k|=\lim_{\epsilon\to 0}(k^2+\epsilon^2)^{\frac{1}{2}}. \tag{2.19}$$

The significance of the various terms in (2.15) to (2.18) is appreciated most easily when $\beta=0$. For, with this value of β, $\lambda=p$ and so (2.15), (2.16) clearly represent the Alfven wave propagation of electric current and vorticity (cf. equations (1.11)). Further, the $e^{-\lambda|x|}$ terms in (2.17) and (2.18) represent the resulting response of the fluid flow and magnetic field, while the $e^{-|k||x|}$ terms define the potential disturbance required to satisfy the governing equations and the boundary condition $\chi = 0$ on $x = 0$. For $\beta \neq 0$, the effects of diffusion are included in the terms $e^{-\lambda|x|}$. However the significance of the $e^{-|k||x|}$ terms is still the same.

Defining

$$\bar{f}(x,k,p) = -ik\frac{1+\beta p}{k^2-p^2}\left\{\frac{p}{|k|}e^{-|k||x|} - e^{-p|x|}\right\}, \quad (2.20)$$

$$\bar{g}(x,k,p) = ik\frac{1+\beta p}{k^2-p^2}\left\{\frac{\lambda}{p}e^{-\lambda|x|} - e^{-p|x|}\right\}, \quad (2.21)$$

equation (2.17) after multiplication by $-ik$ becomes

$$(\overline{\partial\psi/\partial y})(x,k,p) = \bar{f}(x,k,p)\,\bar{j}_0(k,p) + \bar{g}(x,k,p)\,\bar{j}_0(k,p). \quad (2.22)$$

Inverting this expression with respect to k leads to the convolution integrals

$$\widehat{\frac{\partial\psi}{\partial y}}(x,y,p) = \int_{-\infty}^{\infty}\hat{f}(x,y-\xi,p)\hat{j}_0(\xi,p)\,d\xi + \int_{-\infty}^{\infty}\hat{g}(x,y-\xi,p)\hat{j}_0(\xi,p)\,d\xi. \quad (2.23)$$

The convenience of this representation is evident when $\beta = 0$. In this case $g = 0$ and so we are left only with the first integral in (2.23). Now, as $|x| \to 0$,

$$\hat{f}(x,y,p) \to (\operatorname{sgn} y)(1+\beta p)\,F(p|y|), \quad (2.24)$$

$$\hat{g}(x,y,p) \to (\operatorname{sgn} y)(1+\beta p)^{\frac{1}{2}}\,G((p/\beta)^{\frac{1}{2}}|y|, \beta p) \quad (y \neq 0), \quad (2.25)$$

where

$$F(x) = (1/\pi)\{\sin x \operatorname{Ci}(x) - \cos x \operatorname{si}(x)\}, \quad (2.26)$$

$$G(x,s) = \frac{1}{\pi}\int_1^{\infty}\frac{z(z^2-1)^{\frac{1}{2}}}{z^2+s}e^{-zx}\,dz, \quad (2.27)$$

for $\operatorname{Re} x > 0$, $\operatorname{Re} s > 0$. (The cosine and sine integrals $\operatorname{Ci}(x)$ and $\operatorname{si}(x)$ are defined in Erdelyi *et al.* (1953, II, § 9.8).) The limit (2.24) may be obtained without difficulty by putting $x = 0$ and rotating the contour of integration into the line $\operatorname{Im}(k/p) = 0$. The inverse of $\bar{g}$, for $y > 0$, is obtained by deforming the contour around the cut $\operatorname{Re}(k/p^{\frac{1}{2}}) = 0$, $\operatorname{Im}(\beta^{\frac{1}{2}}k/p^{\frac{1}{2}} + i) < 0$ (provided $0 < x \ll y$) and proceeding to the limit $|x| \to 0$. It follows that, on $x = 0$,

$$\hat{u}(y,p) = (1+\beta p)\int_{-1}^{1}\hat{j}_0(\xi,p)\operatorname{sgn}(y-\xi)\,F(p|y-\xi|)\,d\xi$$
$$+ (1+\beta p)^{\frac{1}{2}}P\int_{-1}^{1}\hat{j}_0(\xi,p)\operatorname{sgn}(y-\xi)\,G((p/\beta)^{\frac{1}{2}}|y-\xi|, \beta p)\,d\xi, \quad (2.28)$$

where

$$u(y,t) = (\partial\psi/\partial y)(0,y,t). \quad (2.29)$$

Since

$$\hat{u}(y,p) = -(1/p) \quad (|y| < 1), \quad (2.30)$$

we are left with an integral equation for $\hat{j}_0(y,p)$ to be solved for $|y| < 1$.

Hence if the complete solution of this equation were known, the values of ψ and χ, etc., would be expressible in integral form for all x, y and t. However, no general solution has been obtained owing to the complicated form of (2.28). Instead approximate solutions valid in certain circumstances are obtained by two methods, namely by approximating (2.28) itself (§§ 4 and 6) or by solving (2.22) with $x = 0$ by Wiener–Hopf methods (§§ 3 and 5). In this way approximate values of $\partial\psi/\partial y$ and j are obtained in certain regions of space, valid for particular time scales.

Finally, since the inverse Fourier transform of (2.16) may be determined exactly, it is appropriate to state the inverse in this section. The inversion of $e^{-\lambda|x|}$ is determined (Erdelyi *et al.* 1954, p. 16, equation (26)) and hence $\hat{\jmath}(x, y, p)$ can be expressed as the convolution integral

$$\hat{\jmath}(x,y,p) = \frac{px}{\pi}\int_{-1}^{1} \hat{\jmath}_0(\xi,p)\,\frac{K_1\,[(p/\beta)\,(y-\xi)^2+(p^2x^2/(1+\beta p))]^{\frac{1}{2}}}{[(1+\beta p)\,(y-\xi)^2+\beta px^2]^{\frac{1}{2}}}\,d\xi, \tag{2.31}$$

where $K_\nu(x)$ is the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\nu\pi}. \tag{2.32}$$

3. The initial motion $\beta = 0$, $t \ll 1$

Since the assumption $\beta = 0$ introduces a considerable simplification to the governing equations, this assumption is made here and in § 4. It is a good approximation over the time scales involved except in certain regions, especially those in which the electric current and vorticity are described by δ-functions. The strength of the δ-functions must be regarded as the limit as $\beta \to 0$ of integral constraints valid over a small interval whose width is $O(\beta t)^{\frac{1}{2}}$ (see § 5).

(a) *The stagnation-point flow*

The initial motion at $t = 0+$ near $z = 0$ $(x > 0)$ is given by

$$\left.\begin{aligned}\psi_0(x,y) &= \tfrac{1}{2}xy\\ \chi_0(x,y) &= 0\end{aligned}\right\}\quad (|z| \ll 1), \tag{3.1}$$

where axes are taken moving with the plate. Subsequently it is clear by inspection of the governing equations (1.8) to (1.10) and the boundary conditions on ψ and χ on the plate, that the flow (and perturbation magnetic field) are described by

$$\psi(x,y,t) = \begin{Bmatrix} 0 & (x<t),\\ \tfrac{1}{2}(x-t)\,y & (x>t),\end{Bmatrix} \tag{3.2}$$

and

$$\chi(x,y,t) = \begin{Bmatrix} \tfrac{1}{2}xy & (x<t),\\ \tfrac{1}{2}yt & (x>t),\end{Bmatrix} \tag{3.3}$$

provided $|z| \ll 1$, $t \ll 1$.

These equations give a simple description of the motion and provide a good (yet non-trivial) illustration of the effect 'of freezing the magnetic field lines' into the flow. Equation (3.2) shows that the flow is stopped completely by the current–vortex sheet

$$j(x,y,t) = -\omega(x,y,t) = \tfrac{1}{2}y\delta(x-t), \tag{3.4}$$

while the flow pattern is propagated downstream unaltered. The total magnetic field, given in the form

$$\mathbf{B} = \mathbf{i}+A\mathbf{b}, \tag{3.5}$$

is

$$\mathbf{B} = \begin{Bmatrix} (1+\tfrac{1}{2}Ax,\ -\tfrac{1}{2}Ay) & (x<t),\\ (1+\tfrac{1}{2}At,\ 0) & (x>t),\end{Bmatrix} \tag{3.6}$$

indicating that the magnetic field ahead of the current-vortex sheet is being uniformly compressed but remains unaltered once the sheet has passed (see figure 3).

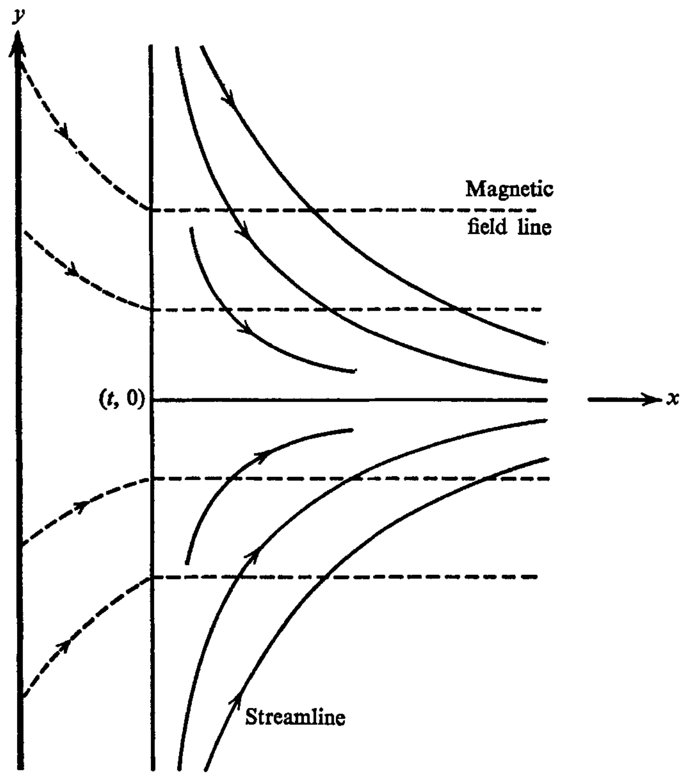

FIGURE 3. The instantaneous streamlines and magnetic field near the stagnation point ($t \ll 1$).

(b) *The flow near the end of the plate* ($y = -1$)

In order to discuss the initial motion in the neighbourhood of $(0, -1)$ the length scale is altered and co-ordinates are taken with the origin now at the end of the plate. Hence by making the transformations

$$y = \tfrac{1}{2}y^* - 1, \quad x = \tfrac{1}{2}x^*, \tag{3.7}$$

the initial conditions for the flow near $(0, 0)$, for axes moving with the plate, become (dropping the star)

$$\psi_0(x, y) = \operatorname{Im}(-iz)^{\frac{1}{2}}, \quad \chi_0(x, y) = 0 \quad (|z| \ll 1). \tag{3.8}$$

The plate is now regarded as semi-finite so that the problem is to find ψ and χ subject to the initial conditions (3.8) and the boundary conditions

$$\left.\begin{aligned} &\chi = 0, \\ &\psi = 0 \quad (y > 0), \quad j = 0 \quad (y < 0), \end{aligned}\right\} \quad \text{on} \quad x = 0, \tag{3.9}$$

and
$$\left.\begin{aligned}\psi &\sim \operatorname{Im}(-iz)^{\frac{1}{2}},\\ \chi &\sim 0,\end{aligned}\right\} \quad \text{as} \quad \frac{|z|}{t}\to\infty.\dagger \tag{3.10}$$

Since the method of solution is somewhat lengthy, it is omitted here but is presented in detail in appendix A. The vorticity (and electric current) are obtained in the form (illustrated in figure 4)

$$\left.\begin{aligned}\omega &= \frac{1}{2}\left\{(t-|x|)^{-\frac{1}{2}}\,\delta(y)+y^{-\frac{3}{2}}F_1\left(\frac{t-|x|}{y}\right)\right\},\\ \text{for}\quad |x| &\leqslant t,\ y\geqslant 0 \quad\text{and}\quad \omega = 0 \text{ otherwise,}\end{aligned}\right\} \tag{3.11}$$

where $F_1(\tau)$ is given by (A 17). Moreover, the fluid velocity and perturbation magnetic field are now in principle known, though interpretation of the integrals

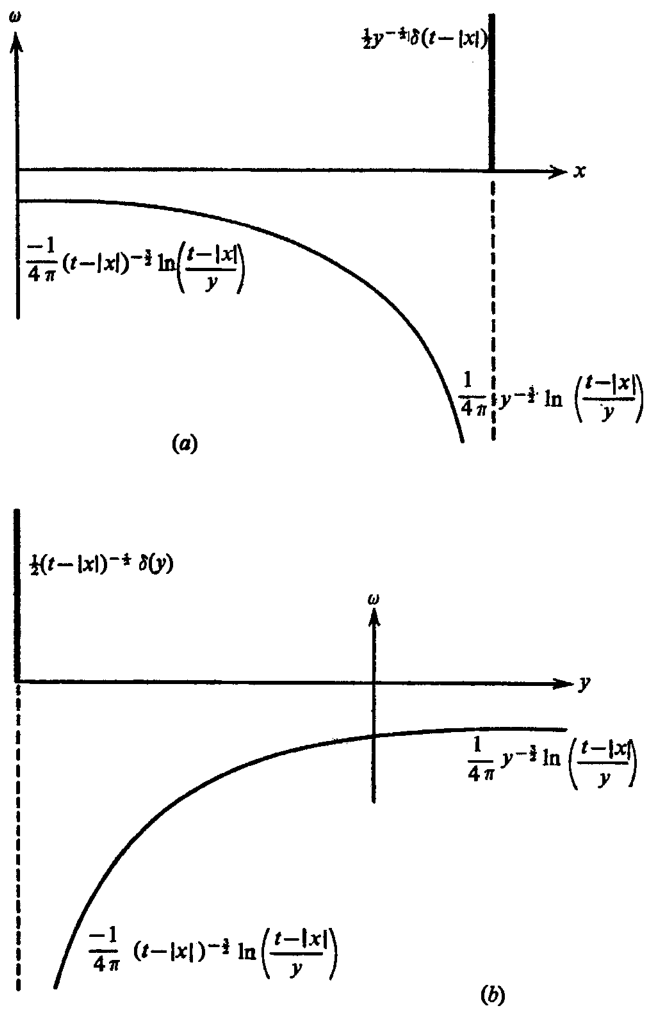

FIGURE 4. The vorticity ω plotted (a) against x, for fixed y ($t/y \gg 1$) and (b) against y, for fixed x.

† This boundary condition reduces to the initial condition (3.8) as $t\to 0$.

by which they may be expressed appears formidable. However, on $x = 0$, the velocity distribution is given by

$$\frac{\partial \psi}{\partial y}(0, y, t) = \begin{cases} 0 & (y \geqslant 0), \\ \tfrac{1}{2}(-y)^{-\frac{1}{2}} G_1\left(-\dfrac{t}{y}\right) & (y < 0), \end{cases} \tag{3.12}$$

where $G_1(\tau)$ is given by (A 21).

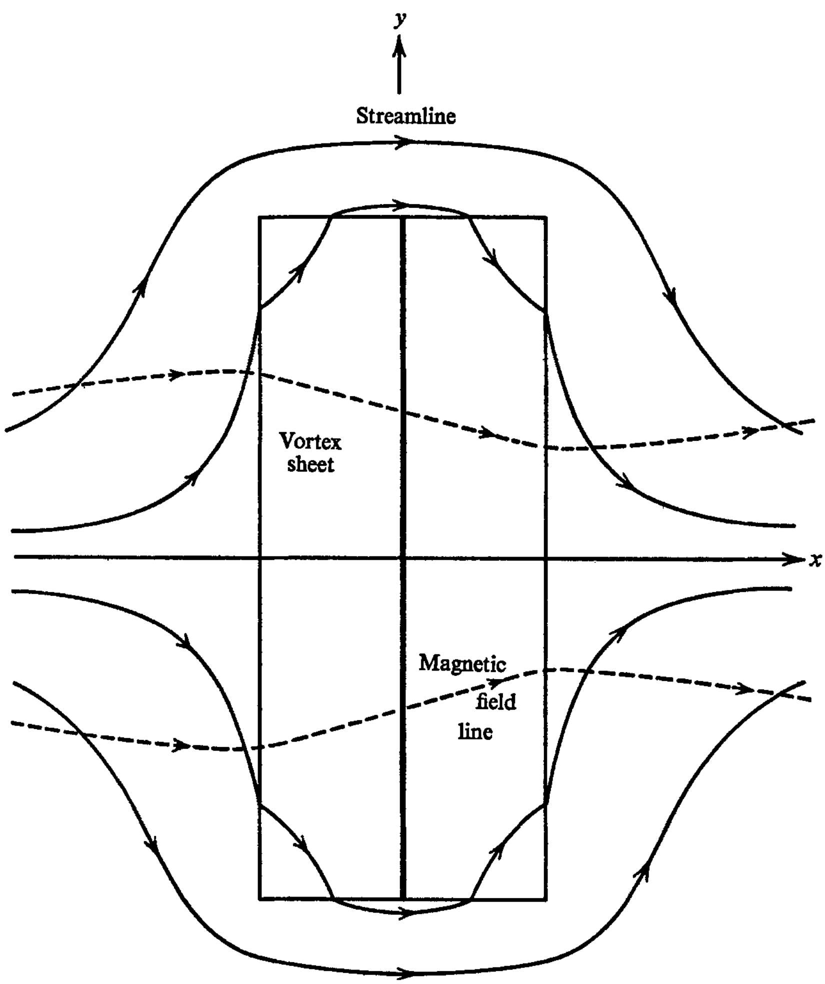

FIGURE 5. The instantaneous streamlines and magnetic field ($t \ll 1$) for axes moving with the plate.

The solution indicates the existence of two vortex sheets. First, there is the initial discontinuity which propagates from both sides of the plate, namely

$$(\operatorname{sgn} x)\, j(x, y, t) = -\omega(x, y, t) = -\tfrac{1}{2} y^{-\frac{1}{2}}\, \delta(|x| - t). \tag{3.13}$$

Secondly, vorticity (and current) propagate from the end of the plate with strength

$$j_0(y, t) = -\tfrac{1}{2} t^{-\frac{1}{2}}\, \delta(y), \tag{3.14}$$

corresponding to the discontinuity of the horizontal velocity at $y = 0$; in particular (3.12) indicates that

$$(\partial\psi/\partial y)(0,y,t) \sim \tfrac{1}{2}t^{-\frac{1}{2}} \quad (0 < -y \ll t). \tag{3.15}$$

The uniformity of the horizontal velocity is not indicative of the propagation of waves in the y direction but results from the length scale of the potential disturbance being independent of direction.

Elsewhere the propagation of vorticity is continuous and is of opposite sign to that in the sheets. This vorticity is clearly such as to inhibit motion of the fluid across the magnetic field lines (see figure 5). Near the sheets the vorticity increases logarithmically and is given (see equation (A 22)) by

$$\left.\begin{aligned} \omega &\sim \frac{1}{4\pi}y^{-\frac{3}{2}}\ln\left(\frac{t-|x|}{y}\right) && (t-|x| \ll y),\\ \text{and} \quad \omega &\sim \frac{-1}{4\pi}(t-|x|)^{-\frac{3}{2}}\ln\left(\frac{t-|x|}{y}\right) && (y \ll t-|x|). \end{aligned}\right\} \tag{3.16}$$

The curious symmetry of the vorticity in the neighbourhood of the sheets and in the sheets themselves does not extend into the main body of the flow. Finally, it is worth noting that the total vorticity (and current) propagating away from either side of the plate is zero for $t > 0$ (equation (A 19)).

(c) *The flow past a finite plate*

The nature of the magnetic field and velocity distribution near the middle and ends of the plate combine quite simply to give a general indication of the initial motion. In particular the fluid is brought to rest near the middle while a slight 'leaking' is present at the edges of the plate where the horizontal velocity is decaying as $t^{-\frac{1}{2}}$ (figure 5).

4. The asymptotic solution $\beta = 0$, $t \to \infty$

Supposing that the fluid is perfectly conducting, the nature of the flow can be determined everywhere (for $t \geqslant 0$), provided the solution to the integral equation

$$\int_{-1}^{1} \hat{j}_0(\xi,p)\,\mathrm{sgn}\,(y-\xi)\,F(p\,|y-\xi|)\,d\xi = \frac{-1}{p} \quad (\mathrm{Re}\,p > 0), \tag{4.1}$$

(equation (2.28) with $\beta = 0$) is known. No general solution of this equation has been obtained. However, if we assume that the solution does not oscillate or grow exponentially with time, the asymptotic motion ($t \to \infty$) may be found. Stewartson (1956) was obliged to make similar assumptions when determining the transient motion past a sphere and the reasons he gives apply equally well to the present problem. Moreover, since this assumption was verified for the initial motion past the plate ($t \ll 1$), this provides yet further evidence that it is correct.

Consider the inversion integral

$$j_0(y,t) = \int_C \hat{j}_0(y,p)\,e^{pt}\,dp, \tag{4.2}$$

where $\hat{j}_0(y,p)$ only has singularities when $\operatorname{Re} p < 0$, $p = 0$. (Instabilities correspond to singularities of $\hat{j}_0(y,p)$ in the right-hand half plane.) Since the contour of integration may be deformed into the negative half plane around the singularities, it follows that the dominant contribution to the inversion integral, as $t \to \infty$, is obtained when

$$p = O(t^{-1}). \tag{4.3}$$

The asymptotic behaviour of $F(x)$ may be determined (Erdelyi *et al.* 1953, II, p. 146, equations (8) to (11)) as

$$F(x) = \begin{cases} \dfrac{1}{2} + \dfrac{x \ln x}{\pi} + \dfrac{\gamma - 1}{\pi} x - \dfrac{x^2}{4} + O(x^3 \ln x) & (x \ll 1), \\ \dfrac{1}{\pi x} + O\left(\dfrac{1}{x^3}\right) & (x \gg 1), \end{cases} \tag{4.4}$$

where γ is Euler's constant. Thus integrating (4.1) by parts leads to

$$\hat{J}(y,p) + p \int_{-1}^{1} \hat{J}(\xi,p) F'(p|y-\xi|)\, d\xi = -(1/p), \tag{4.5}$$

where

$$J(y,t) = \int_{-1}^{y} j_0(\xi,t)\, d\xi, \tag{4.6}$$

and

$$J(-1,t) = J(1,t) = 0, \tag{4.7}$$

due to the skew-symmetry of $j_0(y,t)$. This form of the equation has two advantages. First, the current sheets that propagate from $y = \pm 1$ are now represented by the non-zero values of $J(\pm 1, t)$ rather than by δ-functions as they appear in $j_0(y,t)$. Secondly, the equation is now in a convenient form for solution by successive approximation. Thus substituting

$$\hat{J}(y,p) = \hat{J}_0(y,p) + \hat{J}_1(y,p) + \ldots, \tag{4.8}$$

into (4.5) we obtain

$$\hat{J}_0(y,p) = -1/p, \tag{4.9}$$

$$\begin{aligned} \hat{J}_1(y,p) &= -\frac{p}{\pi} \int_{-1}^{1} \hat{J}_0(\xi,p) \{\gamma + \ln(p|y-\xi|)\}\, d\xi, \\ &= \frac{1}{\pi} \{2\gamma + (1-y)[\ln p(1-y) - 1] + (1+y)[\ln p(1+y) - 1]\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \hat{J}_2(y,p) &= \frac{-p}{\pi} \int_{-1}^{1} [\hat{J}_1(\xi,p)\{\gamma + \ln(p|y-\xi|)\} - \tfrac{1}{2}\pi \hat{J}_0(\xi,p) p|y-\xi|]\, d\xi, \\ &= -\frac{2}{\pi^2} p \ln p \{(4\gamma - 5 + 2\ln(2p)) + (1-y)\ln(1-y) \\ &\qquad + (1+y)\ln(1+y)\} + pQ(y), \end{aligned} \tag{4.11}$$

where $Q(y)$ is a function of y that need not be determined and $\hat{J}_n(y,p)$, for $n > 2$, is defined in a similar manner. In this way $\hat{J}_n \ll \hat{J}_{n-1}$ uniformly for $y \in [-1, 1]$.

Asymptotically, as $t \to \infty$, the only terms contributing to the inversion of $\hat{J}(y,p)$ are those singular at $p = 0$. Thus inverting (4.9) to (4.11) and differentiating with respect to y lead to

$$\begin{aligned} j_0(y,t) = &\left\{1 + \frac{2}{\pi t} - \frac{4\ln(t^2/2)}{\pi^2 t^2} - \frac{2}{\pi^2 t^2}\right\} \{\delta(1-y) - \delta(1+y)\} \\ &- \frac{2}{\pi^2 t^2} \ln\left(\frac{1+y}{1-y}\right) + o(t^{-2}), \end{aligned} \tag{4.12}$$

as $t \to \infty$, and so determine the asymptotic generation of vorticity and electric current on the plate.

The ultimate flow is precisely that described by Ludford & Leibovich (1965). Briefly, relative to axes moving with the fluid at infinity, a column of fluid of length $2t$ moves with the plate bounded by a vortex sheet. Outside the column the flow is potential, $O(t^{-1})$, and corresponds to a source of fluid at $(-t, 0)$ and a sink at $(t, 0)$. In the column the magnetic field is increased by $(1, 0)$ for $x > 0$, decreased by $(-1, 0)$ for $x < 0$ and is bounded by current sheets at $y = \pm 1$. Outside the perturbation field is $O(1)$ and corresponds to sources at $(\pm t, 0)$ together with a sink of twice their strength at the origin.

Finally, the nature of the current propagation from the plate when $|y| < 1$ $(t \gg 1)$ is worth noting. In particular it decays very fast, $O(t^{-2})$, and displays the logarithmic behaviour near $y = \pm 1$, similar to that found for the initial motion $(t \ll 1)$.

5. The asymptotic solution $(\beta t)^{\frac{1}{2}} \ll 1 \ll t$

So far no attempt has been made to determine the effects of the magnetic diffusivity $(\beta \neq 0)$. Evidently the current sheets propagating from the ends of the plate will diffuse transversely with a width $O(\beta t)^{\frac{1}{2}}$. Moreover, it may be anticipated that the motion in the vicinity of the current sheet at $y = -1$ will be uninfluenced by the finite width of the plate. Hence axes are taken at the lower end of the plate which is now supposed semi-infinite. This approximation is justified *a posteriori* by the nature of the resulting flow. In particular, for $0 < 1 - |x|/t = O(1)$, we find that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial y} &= -1, \quad \frac{\partial \chi}{\partial y} = \operatorname{sgn} x \quad \text{when} \quad y \gg (\beta t)^{\frac{1}{2}}, \\ \frac{\partial \psi}{\partial y} &= 0, \quad \frac{\partial \chi}{\partial y} = 0 \quad \text{when} \quad (-y) \gg (\beta t)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.1}$$

Thus, for $y = O(\beta t)^{\frac{1}{2}}$, $0 < 1 - |x|/t = O(1)$ the solution of the semi-infinite plate problem gives (with a suitable change of axes) $\partial\psi/\partial y$, $\partial\chi/\partial y$, etc. for the finite plate problem in the region

$$y + 1 = O(\beta t)^{\frac{1}{2}}, \quad 0 < 1 - \frac{|x|}{t} = O(1), \tag{5.2}$$

during the period
$$(\beta t)^{\frac{1}{2}} \ll 1 \ll t, \tag{5.3}$$

where the inequality $(\beta t)^{\frac{1}{2}} \ll 1$ is imposed by the condition that the length scale of diffusion is small compared to the width of the plate.

Since the plate is now assumed semi-infinite it is convenient to consider the transformed integral equation in the form (2.22) with $x = 0$. The problem is now of Wiener–Hopf type which may be solved by decomposition of the kernel

$$ik \frac{1 + \beta p}{k^2 - p^2} \left\{ \frac{\lambda}{p} - \frac{p}{|k|} \right\}. \tag{5.4}$$

The decomposition is not obtained for all k, p. However, it is shown in appendix B that the solution does not oscillate or grow exponentially with time and that the dominant contribution to the inversions of $\bar{u}(k,p)$, $\bar{j}_0(k,p)$ are obtained when

$$p = O(t^{-1}), \quad k = O(\beta t)^{-\frac{1}{2}}. \tag{5.5}$$

Making these approximations a splitting of the kernel is obtained which leads to the solutions

$$\bar{j}_0(k,p) = -p^{-1}(1-iK)^{-\frac{1}{2}}, \tag{5.6}$$

$$\bar{u}(k,p) = i\beta^{\frac{1}{2}}p^{-\frac{3}{2}}\{1-(1+iK)^{\frac{1}{2}}\}/K, \tag{5.7}$$

where

$$K = \beta^{\frac{1}{2}}p^{-\frac{1}{2}}k. \tag{5.8}$$

The k-inversions are performed by deforming the contour of integration around the cut $\operatorname{Re} K = 0$, $\operatorname{Re}(1-iK) < 0$ for $\bar{j}_0$ and around the cut $\operatorname{Re} K = 0$, $\operatorname{Re}(1+iK) < 0$ for $\bar{u}$ and lead to

$$\hat{j}_0(y,p) = -\pi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}y^{-\frac{1}{2}}p^{-\frac{3}{4}}\exp(-p^{\frac{1}{2}}y/\beta^{\frac{1}{2}}) \quad (y > 0), \tag{5.9}$$

$$\hat{u}(y,p) = \frac{1}{\pi p}\int_0^\infty \frac{\eta^{\frac{1}{2}}\exp\{-(1+\eta)(p^{\frac{1}{2}}|y|/\beta^{\frac{1}{2}})\}}{1+\eta}\,d\eta \quad (y < 0). \tag{5.10}$$

The p-inversions are obtained immediately (Erdelyi *et al.* 1954, p. 246, equation (9) and p. 386) in the form

$$j_0(y,t) = \frac{-1}{\pi}(2\pi\beta t)^{-\frac{1}{2}}\exp\left(\frac{-y^2}{8\beta t}\right)K_{\frac{1}{4}}\left(\frac{y^2}{8\beta t}\right) \quad (y > 0), \tag{5.11}$$

$$u(y,t) = \frac{1}{\pi}\int_0^\infty \frac{\eta^{\frac{1}{2}}}{1+\eta}\operatorname{erfc}\left[\left(\frac{y^2}{4\beta t}\right)^{\frac{1}{2}}(1+\eta)\right]d\eta \quad (y < 0), \tag{5.12}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{5.13}$$

Moreover, when $|y| \ll (\beta t)^{\frac{1}{2}}$, the electric current and velocity on $x = 0+$ are

$$j_0(y,t) \sim \frac{-\Gamma(\frac{1}{4})}{2\pi^{\frac{3}{2}}}(\beta t)^{-\frac{1}{2}}\left(\frac{y^2}{4\beta t}\right)^{-\frac{1}{4}}, \tag{5.14}$$

$$u(y,t) \sim \frac{2\Gamma(\frac{3}{4})}{\pi^{\frac{3}{2}}}\left(\frac{y^2}{4\beta t}\right)^{-\frac{1}{4}}, \tag{5.15}$$

while, for $|y| \gg (\beta t)^{\frac{1}{2}}$, both j_0 and u decay exponentially as $\exp(-y^2/4\beta t)$. Finally it is demonstrated in appendix B that the total electric current on the plate and the flux of fluid past the plate are

$$\int_0^\infty j_0(y,t)\,dy = -1, \tag{5.16}$$

$$\int_{-\infty}^0 u(y,t)\,dy = (\beta t/\pi)^{\frac{1}{2}}. \tag{5.17}$$

The integral constraint (5.16) is in agreement with the requirements (5.1), corresponding to a δ-function for the electric current when $\beta = 0$. The finite flux of fluid $(\beta t/\pi)^{\frac{1}{2}}$ past the plate is unexpected but presumably an exact analysis would reveal the return of this small quantity of fluid outside the layer.

Since (5.9) is only approximate it is unnecessary to retain the full accuracy of (2.31). Thus after one minor approximation $1+\beta p \sim 1$, (5.9) and (2.31) lead to

$$\hat{j}(x,y,p) = \frac{-p^{\frac{1}{4}}x}{\pi^{\frac{3}{2}}\beta^{\frac{1}{4}}}\int_0^\infty \frac{\exp\left(-\frac{p^{\frac{1}{2}}\xi}{\beta^{\frac{1}{2}}}\right)}{\xi^{\frac{1}{2}}}\frac{K_1\left[\frac{p}{\beta}(y-\xi)^2+p^2x^2\right]^{\frac{1}{2}}}{[(y-\xi)^2+\beta p x^2]^{\frac{1}{2}}}\,d\xi. \quad (5.18)$$

On $y = 0$, the current distribution may now be obtained without further approximation. Making the substitution $\xi = (\beta p)^{\frac{1}{2}}\eta$, the inverse of (5.18) is determined (Erdelyi *et al.* 1954, p. 277, equation (10)) and leads after some manipulation to

$$j(x,0,t) = -(\operatorname{sgn} x)(2\pi\beta|x|)^{-\frac{1}{2}} \quad (0 < 1-(|x|/t) = O(1)).\dagger \quad (5.19)$$

For both x and y non-zero the p-inversion of (5.18) has not been obtained. However, provided $|y|/\surd(\beta t) \ll 1$ and $|x|/t \ll 1$, (5.18) may be approximated further by retaining only the first term in the expansion of the exponential and Bessel functions. The resulting expression is inverted with respect to p giving a convolution integral. In terms of a similarity variable the electric current is

$$\left.\begin{aligned} & j(x,y,t) \sim \frac{-(\operatorname{sgn} x)}{(2\pi\beta|x|)^{\frac{1}{2}}}F_2\left(\frac{y/\surd(\beta t)}{|x|/t}\right)\begin{cases}(|y| \ll \surd(\beta t)),\\(|x| \ll t),\end{cases}\\ \text{where}\quad & F_2(\theta) = \frac{\Gamma(\frac{3}{4})}{\pi^2}\int_{\phi=0}^\infty\int_{\tau=0}^1 \frac{\exp[-(\theta-\phi)^2\tau]}{\phi^{\frac{1}{2}}(1-\tau)^{\frac{3}{4}}}\,d\tau\,d\phi,\\ \text{and}\quad & F_2(\theta) \sim \frac{\Gamma(\frac{1}{4})}{\pi}\theta^{-\frac{1}{2}} \quad \text{as} \quad \theta\to\infty,\\ & F_2(0) = 1,\\ & F_2(\theta) \sim \frac{\Gamma(\frac{3}{4})}{2\pi}(-\theta)^{-\frac{3}{2}} \quad \text{as} \quad \theta\to-\infty. \end{aligned}\right\} \quad (5.20)$$

At first sight, the large velocities near $x = y = 0$ may seem somewhat surprising, for it could be argued that the introduction of dissipation in the equations would smooth out discontinuities, and certainly not increase the velocity of the fluid. However, the effect of the dissipative term is rather indirect acting through the diffusion of electric current. The physical mechanisms involved may be understood by determining which terms of equations (1.8) to (1.10) are important.

Evidently the crucial step in the decomposition of the kernel (5.4) is the neglect of the second term. This corresponds to a neglect of the potential disturbance of the velocity on $x = 0$. Though we may expect the principal contribution to the potential disturbance of the velocity to be on length scales $O(t)$ it is curious that it is not of some importance to the immediate vicinity of the end of the plate on length scales $O(\beta t)^{\frac{1}{2}}$. Moreover, if the dominant contribution to the inversion integral (2.22) in the region $0 < 1-|x|/t = O(1)$, $y = O(\beta t)^{\frac{1}{2}}$ is obtained when $p = O(t^{-1})$, $K = O(1)$, then consideration of the various terms in (2.17) indicates that

$$\overline{(\partial\psi/\partial y)}(x,k,p) = -i\beta^{\frac{1}{2}}p^{-\frac{3}{2}}(K^{-1})_\oplus(1+iK)^{\frac{1}{2}}\exp[-p(1+K^2)^{\frac{1}{2}}|x|]. \quad (5.21)$$

† This result is perhaps obtained more easily by following the same procedure leading to equation (5.22).

On $y = 0$, the inversion leads without difficulty to

$$\frac{\partial\psi}{\partial y}(x,0,t) = \frac{-i}{2\pi}\int_{-(t^2/x^2-1)^{\frac{1}{2}}}^{(t^2/x^2-1)^{\frac{1}{2}}} \frac{(1+iK)^{\frac{1}{2}}}{K_{\oplus}}\,dK,$$

$$= \frac{\sqrt{2}}{\pi}\left\{\left(\frac{t}{|x|}-1\right)^{\frac{1}{2}} - \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{1}{2}\left(\frac{t}{|x|}-1\right)\right]^{\frac{1}{2}}\right\} - \frac{1}{2}. \tag{5.22}$$

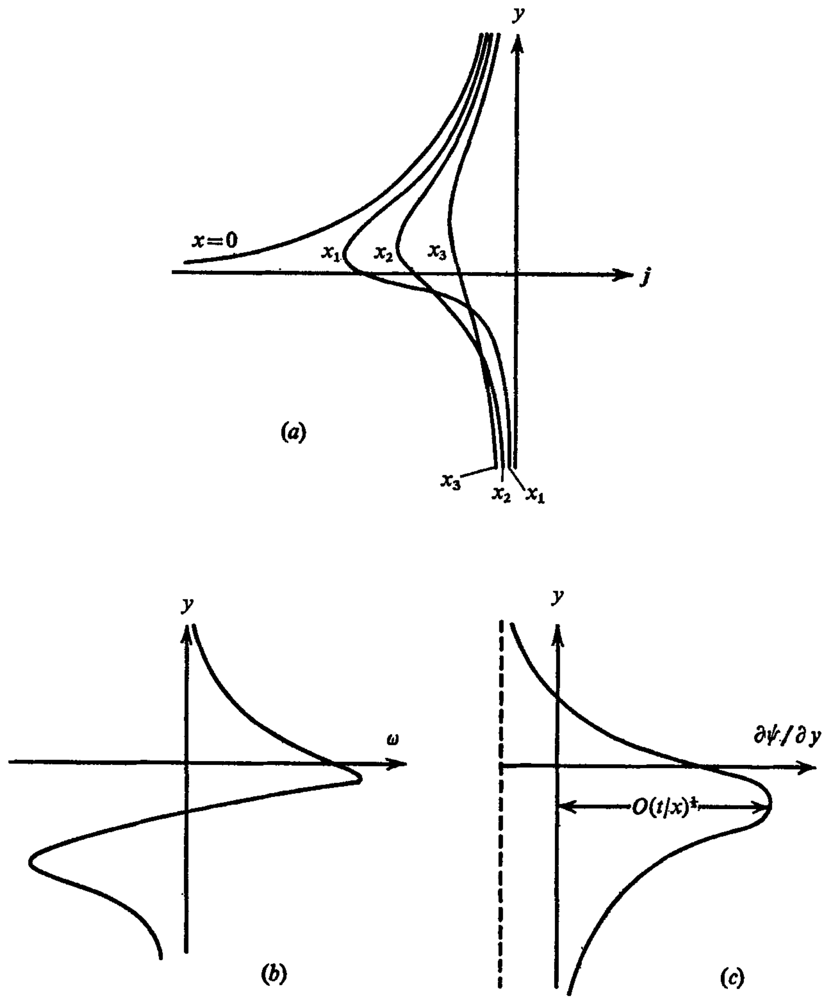

FIGURE 6. The profiles of (a) the electric current, (b) the vorticity and (c) the velocity, for fixed x_i ($x_3 > x_2 > x_1 > 0$) and t, when $y = O(\beta t)^{\frac{1}{2}}$.

Since the velocity is $O(1)$ when $|x|/t = O(1)$ the assumption that (5.21) gives the dominant contribution is suspect. However, it is clearly correct asymptotically

$$\frac{\partial\psi}{\partial y}(x,0,t) \sim \frac{\sqrt{2}}{\pi}\left(\frac{t}{|x|}\right)^{\frac{1}{2}} \quad (|x| \ll t), \tag{5.23}$$

and indicates that the large horizontal velocities are confined to the region $|y|/\sqrt{(\beta t)} \ll 1$, $|x|/t \ll 1$, while any jet-like structure completely disappears as $|x|/t \to 1$.

The total electric current in the sheet is propagated as an Alfven wave subject to the constraint $\int_{-\infty}^{\infty} j\,dy = -1$. Once clear of the plate ($(\beta t)^{\frac{1}{2}} \ll |x|$) it spreads transversely by diffusion subject to the approximate equations

$$\left.\begin{aligned} \frac{\partial \omega}{\partial t} = \frac{\partial j}{\partial x}, \quad \frac{\partial \chi}{\partial t} = \frac{\partial \psi}{\partial x} + \beta \frac{\partial^2 \chi}{\partial y^2}, \\ \omega = -\frac{\partial^2 \psi}{\partial y^2}, \quad j = -\frac{\partial^2 \chi}{\partial y^2}. \end{aligned}\right\} \tag{5.24}$$

Evidently for $y/\surd(\beta t) \ll 1$, $|x|/t \ll 1$ the electrostatic approximation is valid (neglect of $\partial\chi/\partial t$ in (5.24)) and this may be traced through the similarity variable θ in (5.20).

There appears to be no simple explanation for the large velocities in the region $y/\surd(\beta t) \ll 1$, $|x|/t \ll 1$. The nature of the electric current distribution (5.20) depends on the approximate equations (5.24) (with $\partial\chi/\partial t$ neglected) together with the boundary conditions on $x = 0$, $y > 0$ and is illustrated in figure 6(*a*). Now for $x > 0$, $y = 0$ we have $\partial j/\partial x > 0$ implying $\partial\omega/\partial t > 0$, while for $x = 0$, $(-\theta) \gg 1$ we have $\partial j/\partial x < 0$ implying $\partial \omega/\partial t < 0$. Thus the vorticity develops a dipole-like structure of increasing strength and so the fluid is accelerated by the action of the $\mathbf{j} \times \mathbf{B}$ force (figures 6(*b*) and (*c*)).

6. The final development of the motion $\beta t \to \infty$

We conclude this paper with some remarks about the final development of the motion $(\beta t)^{\frac{1}{2}} \to \infty$. Vorticity and electric current penetrate a distance $O(t)$ in the x direction as a result of Alfven wave propagation and a distance $O(\beta t)^{\frac{1}{2}}$ in the y direction due to magnetic diffusion. Restricting attention to the region $0 < 1 - |x|/t = O(1)$ an approximate expression for the electric current is obtained.

Since we are concerned with time scales large compared to β^{-1} the dominant contribution to the inversion integral of $\hat{j}_0(y, p)$ is obtained when

$$\beta \gg p. \tag{6.1}$$

Making this approximation, (2.28) becomes

$$P \int_{-1}^{1} \hat{j}_0(\xi, p) \left\{ \frac{\beta^{\frac{1}{2}}}{\pi p^{\frac{1}{2}}} \frac{1}{y - \xi} \right\} d\xi = \frac{-1}{p}, \tag{6·2}$$

where only the largest term in the integrals has been retained. Moreover (6.2) has the symmetric solution

$$\hat{j}_0(y, p) = \frac{1}{(\beta p)^{\frac{1}{2}}} \frac{y}{(1 - y^2)^{\frac{1}{2}}}. \tag{6.3}$$

Applying the minor approximation $1 + \beta p \sim 1$, (6.3) and (2.31) lead to

$$\hat{j}(x, y, p) = \frac{(\operatorname{sgn} x)\, p}{\pi \beta} \int_{-1}^{1} \frac{\xi}{(1-\xi^2)^{\frac{1}{2}}} \frac{K_1[p^2 x^2 + (p/\beta)(y - \xi)^2]^{\frac{1}{2}}}{[p^2 + (p/\beta x^2)(y - \xi)^2]^{\frac{1}{2}}}\, d\xi. \tag{6.4}$$

The p-inversion is determined (Erdelyi *et al.* 1954, p. 284, equation (47)) and leads after some manipulation to

$$j(x,y,t) = \frac{1}{\pi\beta x}\int_{-1}^{1}\frac{\xi}{(1-\xi^2)^{\frac{1}{2}}}\left\{\frac{t}{(t^2-x^2)^{\frac{1}{2}}}\cosh\left[\frac{(y-\xi)^2}{2\beta x^2}(t^2-x^2)^{\frac{1}{2}}\right]\right.$$
$$\left.-\sinh\left[\frac{(y-\xi)^2}{2\beta x^2}(t^2-x^2)^{\frac{1}{2}}\right]\right\}\exp\left[-\frac{(y-\xi)^2}{2\beta x^2}t\right]d\xi. \quad (6.5)$$

When $|x| = O(t)$, the electric current is small $O(\beta t)^{-2}$. However, when $|x|/t \ll 1$ the electric current is considerably larger and is given by

$$j(x,y,t) = \frac{1}{\pi\beta x}\int_{-1}^{1}\frac{\xi}{(1-\xi^2)^{\frac{1}{2}}}\exp\left[-\frac{t}{\beta x^2}(y-\xi)^2\right]d\xi \quad (|x| \ll t). \quad (6.6)$$

Restricting attention to this smaller region $|x| \ll t$, we may determine the velocity by making the further approximation

$$K = \beta^{\frac{1}{2}}p^{-\frac{1}{2}}k \gg 1, \quad (6.7)$$

in the transforms of § 2. Hence (2.22) is given approximately by

$$\overline{\frac{\partial\psi}{\partial y}}(x,k,p) = i\frac{\beta^{\frac{1}{2}}}{p^{\frac{1}{2}}}\frac{|k|}{k}e^{-(\beta p)^{\frac{1}{2}}|k||x|}\bar{j}_0(k,p). \quad (6.8)$$

Inverting this expression with respect to k leads to the convolution integral

$$\widehat{\frac{\partial\psi}{\partial y}}(x,y,p) = \frac{1}{\pi p}\int_{-1}^{1}\frac{\xi}{(1-\xi^2)^{\frac{1}{2}}}\frac{y-\xi}{\beta p x^2+(y-\xi)^2}d\xi, \quad (6.9)$$

and this is readily inverted with respect to p giving

$$\frac{\partial\psi}{\partial y}(x,y,t) = \frac{1}{\pi}\int_{-1}^{1}\frac{\xi}{(1-\xi^2)^{\frac{1}{2}}}\frac{1-\exp\left[-(t/\beta x^2)(y-\xi)^2\right]}{y-\xi}d\xi. \quad (6.10)$$

Provided $|x| \ll (t/\beta)^{\frac{1}{2}}$, the flow described by (6.6) and (6.10) takes a different character in the following three regions (figure 2(*c*)):

Region I, $|y|-1 \gg (\beta x^2/t)^{\frac{1}{2}}$,

$$\left.\begin{aligned}(\partial\psi/\partial y) &\sim -1+|y|/(1-y^2)^{\frac{1}{2}},\\ j &\sim 0.\end{aligned}\right\} \quad (6.11)$$

Region II, $1-|y| \gg (\beta x^2/t)^{\frac{1}{2}}$,

$$\left.\begin{aligned}\frac{\partial\psi}{\partial y} &\sim -1+\left(\frac{\beta}{\pi t}\right)^{\frac{1}{2}}\frac{|x|}{(1-y^2)^{\frac{3}{2}}},\\ j &\sim \frac{\operatorname{sgn} x}{(\pi\beta t)^{\frac{1}{2}}}\frac{y}{(1-y^2)^{\frac{1}{2}}}.\end{aligned}\right\} \quad (6.12)$$

Region III, $1+y = O(\beta x^2/t)^{\frac{1}{2}}$,

$$\left.\begin{aligned}\frac{\partial\psi}{\partial y} &\sim \frac{-1}{\sqrt{2}\,\pi}\left(\frac{t}{\beta x^2}\right)^{\frac{1}{4}}G_0\left[\left(\frac{t}{\beta x^2}\right)^{\frac{1}{2}}(1+y)\right],\\ j &\sim \frac{\operatorname{sgn} x}{\sqrt{2}\,\pi(\beta t)^{\frac{1}{2}}}\left(\frac{t}{\beta x^2}\right)^{\frac{1}{4}}F_0\left[\left(\frac{t}{\beta x^2}\right)^{\frac{1}{2}}(1+y)\right],\end{aligned}\right\} \quad (6.13)$$

where

$$
\left.\begin{aligned}
G_0(\theta) &= \int_0^\infty \frac{1-\exp\left[-(\theta-\phi)^2\right]}{\phi^{\frac{1}{2}}(\theta-\phi)}\,d\phi,\\
F_0(\theta) &= \int_0^\infty \phi^{-\frac{1}{2}}\exp\left[-(\theta-\phi)^2\right]d\phi.
\end{aligned}\right\} \tag{6.14}
$$

A comparison of (6.13) and (5.20) indicates that the mechanisms involved in producing the jet in both cases are essentially the same; though the functional dependence on the similarity variable is different. The resulting flow is comparable to the asymptotic motion described by Stewartson (1956) (with the corrections made by Ludford & Singh 1963).

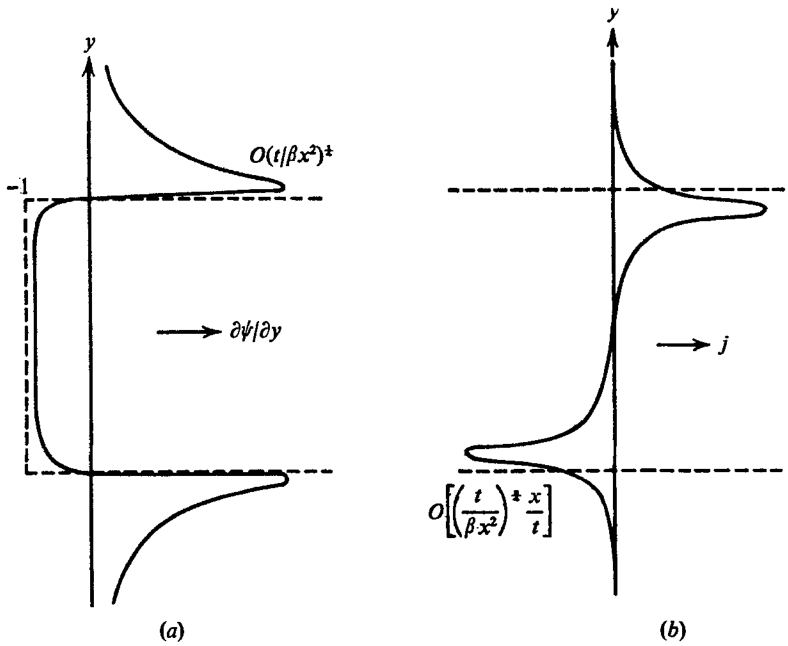

FIGURE 7. The asymptotic (a) velocity and (b) electric current distribution for fixed x, as $(\beta t)^{\frac{1}{2}} \to \infty$.

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Appendix A

The formal solution of the problem presented in § 3 (b) is derived. Equations (1.8) to (1.10) are solved subject to the initial and boundary conditions (3.9) and (3.10).

All the material of § 2 follows except that ψ is replaced by $\psi - y$. In particular (2.22) gives

$$\bar{u}(k,p) = \frac{ik}{(p+|k|)\,|k|}\,\bar{\jmath}_0(k,p). \tag{A 1}$$

Since the boundary conditions (3.9) imply that

$$\bar{u}(k,p) = \int_{-\infty}^{0} \hat{u}(y,p)\,e^{iky}\,dy, \tag{A 2}$$

$$\bar{\jmath}_0(k,p) = \int_0^\infty \hat{\jmath}_0(y,p)\, e^{iky}\, dy, \tag{A 3}$$

the initial velocity distribution

$$u(y,0) = \tfrac{1}{2}(-y)^{-\frac{1}{2}} \quad (y < 0), \tag{A 4}$$

has the Fourier transform

$$\hat{u}(k,0) = \tfrac{1}{2}\sqrt{\pi}\, e^{-i\frac{1}{4}\pi}\, k^{-\frac{1}{2}}. \tag{A 5}$$

It follows by inspection that the condition (3.10) on the velocity is equivalent to the condition

$$\bar{u}(k,p) \sim \tfrac{1}{2}\sqrt{\pi}\, e^{-i\frac{1}{4}\pi} p^{-1} k^{-\frac{1}{2}} \quad \text{as} \quad k/p \to 0, \tag{A 6}$$

where $\bar{u}$ is analytic in the lower half k plane. Moreover, the transform is made analytic in a strip containing the real axis by defining $k^{-\frac{1}{2}}$ as

$$k^{-\frac{1}{2}} = \lim_{\epsilon\to 0} (k - i\epsilon)^{-\frac{1}{2}}, \tag{A 7}$$

which is the natural extension of the definition (2.19).

Evidently (A 1) may be rearranged so that one side of the equation is an analytic function of k in the upper half plane $\operatorname{Im} k > -\epsilon$ and the other side is analytic in the lower overlapping half plane $\operatorname{Im} k < \epsilon$. This is the Wiener–Hopf procedure from which it is argued that both sides of the equation must equal at most a constant. Labelling terms analytic in the upper and lower half planes with the suffices $\oplus$ and $\ominus$ respectively and proceeding to the limit $\epsilon \to 0$, we obtain

$$ip(k^{\frac{1}{2}})_\ominus P_\ominus^{-1}(k,p)\, \bar{u}_\ominus(k,p) = -k(k^{-\frac{1}{2}})_\oplus P_\oplus(k,p) \bar{\jmath}_{0\oplus}(k,p), \tag{A 8}$$

where
$$\left.\begin{aligned} P_\oplus(k,p) &= \left(1+\frac{k}{p}\right)^{-\frac{1}{2}} \exp\left\{\frac{1}{\pi i}\int_0^{(k/p)} \frac{\ln\zeta}{1-\zeta^2} d\zeta\right\}, \\ P_\ominus(k,p) &= \left(1+\frac{k}{p}\right)^{-\frac{1}{2}} \exp\left\{\frac{-1}{\pi i}\int_0^{(k/p)} \frac{\ln\zeta}{1-\zeta^2} d\zeta\right\} \\ &\text{and where } P_\oplus(k,p) \text{ is defined for } -\tfrac{1}{2}\pi < \arg k < \tfrac{3}{2}\pi, \\ &P_\ominus(k,p) \text{ is defined for } -\tfrac{3}{2}\pi < \arg k < \tfrac{1}{2}\pi. \end{aligned}\right\} \tag{A 9†}$$

The constant value (here a function of p) of the two sides of the equation is determined uniquely by the condition on (A 6). It follows that

$$\bar{\jmath}_0(k,p) = -\tfrac{1}{2}\sqrt{\pi}\, e^{i\frac{1}{4}\pi}\, (k^{-\frac{1}{2}})_\oplus P_\oplus^{-1}(k,p), \tag{A 10}$$

$$\bar{u}(k,p) = \tfrac{1}{2}\sqrt{\pi}\, e^{-i\frac{1}{4}\pi} p^{-1} (k^{-\frac{1}{2}})_\ominus P_\ominus(k,p), \tag{A 11}$$

which (by equation (2.22)) is consistent with the condition (3.10).

The inversion of (A 10) is considered in detail. We define the new function $j_1(y,t)$ such that

$$j_0(y,t) = -\tfrac{1}{2}t^{-\frac{1}{2}}\,\delta(y) + j_1(y,t), \tag{A 12}$$

having the double transform

$$\bar{\jmath}_1(k,p) = \bar{\jmath}_0(k,p) + \tfrac{1}{2}\sqrt{\pi}\, p^{-\frac{1}{2}}. \tag{A 13}$$

† The splitting of $\{1+|k|/p\}^{-1}$, when $p = 1$, is given by Carrier, Krook & Pearson (1966, page 396, equation (8.72)).

The function $\bar{j}_1(k,p)$ is inverted with respect to k and leads to

$$\hat{j}_1(y,p) = \begin{cases} -\dfrac{1}{2}\dfrac{p^{\frac{1}{2}}}{\sqrt{\pi}}\displaystyle\int_0^\infty \xi^{-\frac{1}{2}}(1+\xi^2)^{-\frac{1}{4}}\, e^{\rho(\xi)}\, e^{-(py)\xi}\, d\xi & (y > 0), \\ 0 & (y < 0), \end{cases} \tag{A 14}$$

where

$$\rho(\xi) = \frac{-1}{\pi}\int_0^\xi \frac{\ln\zeta}{1+\zeta^2}\,d\zeta, \tag{A 15}$$

provided $\mathrm{Re}\,p > 0$. For $y > 0$, the result is obtained by deforming the contour of integration ($\mathrm{Im}\,k = 0$) around the cut $\mathrm{Re}\,(k/p) = 0$, $\mathrm{Im}\,(k/p) < 0$. (This has been made possible by removal of the constant value of $\bar{j}_0$ as $k \to \infty$). The inverse of

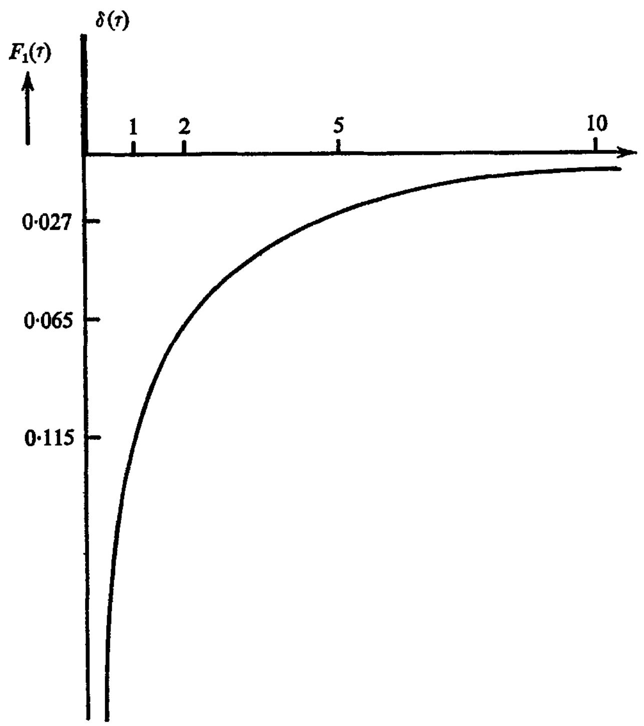

FIGURE 8. The function $F_1(\tau)$ plotted against τ.

$p^{-1}\hat{j}_0(y,p)$ is determined without difficulty as a convolution integral. Hence the electric current distribution on $x = 0+$ is

$$j_0(y,t) = -\tfrac{1}{2}\{t^{-\frac{1}{2}}\,\delta(y) + y^{-\frac{3}{2}}F_1(t/y)\}, \tag{A 16}$$

where

$$\begin{aligned} F_1(\tau) &= \frac{d}{d\tau}\left\{\frac{1}{\pi}\int_0^\tau \frac{e^{-\rho(\xi)}}{(\tau-\xi)^{\frac{1}{2}}\,\xi^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{4}}}\,d\xi\right\}, \\ &= \delta(\tau) + \frac{1}{2\pi\tau}\int_0^\tau \left(\frac{\xi}{\tau-\xi}\right)^{\frac{1}{2}} \frac{2/\pi\ln\xi - \xi}{(1+\xi^2)^{\frac{1}{4}}}\, e^{-\rho(\xi)}\, d\xi, \end{aligned} \tag{A 17}$$

and so by equations (1.11) the vorticity (and electric current) distribution are known everywhere. Moreover the total electric current $J_0(t)$ propagating away from the plate at time $t(>0)$ is

$$J_0(t) = \int_0^\infty \{j_0(y,t) + \tfrac{1}{2}y^{-\frac{1}{2}}\,\delta(t)\}\,dy,$$

$$= [\tilde{j}_0(k,t) + \tfrac{1}{2}\sqrt{\pi}\,e^{i\frac{1}{4}\pi}\,k^{-\frac{1}{2}}\,\delta(t)]_{k=0}. \tag{A 18}$$

Since the Laplace transform of $J_0(t)$ is

$$\hat{J}_0(p) = [-\tfrac{1}{2}\sqrt{\pi}\,e^{i\frac{1}{4}\pi}\,k^{-\frac{1}{2}}\{P_\oplus(k/p)-1\}]_{k=0} = 0 \tag{A 19}$$

the total electric current is zero.

Similarly it can be shown that the horizontal velocity on $x = 0$ is

$$u(y,t) = \begin{cases} 0 & (y>0), \\ \tfrac{1}{2}(-y)^{-\frac{1}{2}}\,G_1(-t/y) & (y<0), \end{cases} \tag{A 20}$$

where

$$G_1(\tau) = \frac{1}{\pi}\int_0^\tau \frac{e^{\rho(\xi)}}{(\tau-\xi)^{\frac{1}{2}}\,\xi^{\frac{1}{2}}(1+\xi^2)^{\frac{3}{4}}}\,d\xi. \tag{A 21}$$

Finally, the asymptotic forms of $F_1(\tau)$ and $G_1(\tau)$ are

$$\left.\begin{aligned} F_1(\tau) &= \begin{cases} \delta(\tau) + (1/2\pi)\ln\tau + O(1) & \text{as } \tau\to 0, \\ -(1/2\pi)\,\tau^{-\frac{3}{2}}\ln\tau + O(\tau^{-\frac{3}{2}}) & \text{as } \tau\to\infty, \end{cases} \\ G_1(\tau) &= \begin{cases} 1-(1/2\pi)\,\tau\ln\tau + O(\tau) & \text{as } \tau\to 0, \\ \tau^{-\frac{1}{2}} + o(\tau^{-\frac{1}{2}}) & \text{as } \tau\to\infty. \end{cases} \end{aligned}\right\} \tag{A 22}$$

The function $F_1(\tau)$ was evaluated on the Cambridge University Titan Computer and was found to converge slowly to its asymptotic forms (figure 8). This is to be expected as the terms neglected are only smaller by a multiple of $(\ln\tau)^{-1}$.

Appendix B

The Wiener–Hopf problem of § 5 is considered; in particular an approximate decomposition of the kernel (5.4) is obtained. In order that the Fourier transform of $\partial\psi(0,y,t)/\partial y$ should exist the boundary condition to the prototype problem is posed in the modified form

$$\frac{\partial\psi}{\partial y}(0,y,t) = \begin{cases} -e^{-\delta y} & (y>0), \\ u(y,t) & (y<0), \end{cases} \tag{B 1}$$

where the limit $\delta\to 0$ is eventually taken.

Taking Fourier and Laplace transforms, (B 1) and (2.22) lead to

$$\frac{ik(1+\beta p)}{k^2-p^2}\left\{\frac{\lambda}{p}-\frac{p}{|k|}\right\}\bar{j}_\oplus(k,p) = \frac{-i}{p(k+i\delta)} + \bar{u}_\ominus(k,p), \tag{B 2}$$

where $|k|$ is defined by equation (2.19), $\ominus$ means the function is analytic for $\operatorname{Im} k < 0$, $\oplus$ means the function is analytic in some overlapping upper half plane and the suffix 0 of j_0 has been dropped.† For convenience we define

$$D(k,p) = \frac{1+\beta p}{k^2-p^2}\left\{\frac{\lambda}{p}-\frac{p}{|k|}\right\}, \tag{B 3}$$

† For the present p is assumed constant.

and 'split' $D(k,p)$ into the product

$$D(k,p) = A_{\oplus}(k,p)\,B_{\ominus}(k,p). \tag{B 4}$$

Thus defining

$$C_{\ominus}(k,p) = kB_{\ominus}(k,p), \tag{B 5}$$

equation (B 2) can be rewritten as

$$A_{\oplus}(k,p)\,\bar{j}_{\oplus}(k,p) + [p(k+i\delta)\,C_{\ominus}(-i\delta,p)]^{-1} = \frac{-1}{p(k+i\delta)}\left\{\frac{1}{C_{\ominus}(k,p)} - \frac{1}{C_{\ominus}(-i\delta,p)}\right\} - \frac{i\bar{u}_{\ominus}(k,p)}{C_{\ominus}(k,p)}, \tag{B 6}$$

where the right-hand side is analytic in the lower half plane and the left-hand side is analytic in some overlapping upper half plane. Hence by conventional Wiener–Hopf arguments both sides are equal to at most a constant which is readily seen to be zero. It follows, in the limit $\delta \to 0$ that,

$$\bar{j}_{\oplus}(k,p) = -\{pkC_{\ominus}(0,p)\,A_{\oplus}(k,p)\}^{-1}, \tag{B 7}$$

$$\bar{u}_{\ominus}(k,p) = \frac{i}{kp}\left\{1 - \frac{C_{\ominus}(k,p)}{C_{\ominus}(0,p)}\right\}. \tag{B 8}$$

Consider the cuts of $D(k,p)$, namely $k = \pm i\epsilon$ and $k = \pm(p/\beta)^{\frac{1}{2}}$. Clearly on continuing $D(k,p)$ analytically with respect to p these cuts remain the same side of the axis $\operatorname{Im} k = 0$, provided $-\pi < \arg p < \pi$. Now continue $A_{\oplus}(k,p)$ and $B_{\ominus}(k,p)$ analytically from some fixed $p\,(\operatorname{Re} p > 0)$ to all p such that $-\pi < \arg p < \pi$. Since the cuts of $A_{\oplus}(k,p)$ and $B_{\ominus}(k,p)$ correspond to the cuts of $D(k,p)$ in the lower and upper half k planes respectively, and since (by the property mentioned above) these cuts remain the same side of the axis $\operatorname{Im} k = 0$, it follows that the analytic continuations of $A_{\oplus}(k,p)$ and $B_{\ominus}(k,p)$ must still be the required 'splitting' of $D(k,p)$. Further, the only zeros and poles of $D(k,p)$ are located at $p = 0,\ -1/\beta$. Hence $A_{\oplus}(k,p)$, $B_{\ominus}(k,p)\,(-\pi < \arg p < \pi)$ are analytic functions of p when $\operatorname{Im} k \geqslant 0$, $\operatorname{Im} k \leqslant 0$ respectively. Since inversions with respect to k are obtained by integrating along $\operatorname{Im} k = 0$, it follows immediately that $\hat{j}_0(y,p)$, $\hat{u}(y,p)$ only have a cut along $\operatorname{Im} p = 0$, $\operatorname{Re} p \leqslant 0$, together with possible zeros or poles at $p = 0,\ -1/\beta$. Hence the solutions do not oscillate or grow exponentially with time and we are justified in assuming that the main contribution to the inversion integrals is obtained when

$$p = O(t^{-1}). \tag{B 9}$$

Since the transverse length scale of the electric current sheet is $O(\beta t)^{\frac{1}{2}}$, the dominant contributions to the inversions of j_0 and u are obtained when

$$k = O(\beta t)^{-\frac{1}{2}}, \tag{B 10}$$

or equivalently, when combined with the approximation (B 9)

$$K = \beta^{\frac{1}{2}}p^{-\frac{1}{2}}k = O(1). \tag{B 11}$$

Making these approximations in $D(k,p)$ leads to

$$D(k,p) \sim \frac{\beta}{p}\frac{(1+K^2)^{\frac{1}{2}}}{K^2}, \tag{B 12}$$

and so by the symmetry of equation (B 3)

$$A_{\oplus}(k,p) \sim \left(\frac{\beta}{p}\right)^{\frac{1}{2}}\frac{(1-iK)^{\frac{1}{2}}}{K}, \quad B_{\ominus}(k,p) \sim \left(\frac{\beta}{p}\right)^{\frac{1}{2}}\frac{(1+iK)^{\frac{1}{2}}}{K}. \tag{B 13}$$

Hence it follows that

$$\bar{j}_{\oplus}(k,p) \sim -p^{-1}(1-iK)^{-\frac{1}{2}}, \qquad \text{(B 14)}$$

$$\bar{u}_{\ominus}(k,p) \sim i\beta^{\frac{1}{2}}p^{-\frac{3}{2}}\{1-(1+iK)^{\frac{1}{2}}\}/K. \qquad \text{(B 15)}$$

Moreover, since

$$\bar{u}_{\ominus}(0,p) = \tfrac{1}{2}\beta^{\frac{1}{2}}p^{-\frac{3}{2}}, \quad \bar{j}_{\ominus}(0,p) = -p^{-1}, \qquad \text{(B 16)}$$

we obtain immediately two integral constraints on the flux of fluid and electric current, namely

$$\int_{-\infty}^{0} u(y,t)\,dy = \tilde{u}(0,t) = (\beta t/\pi)^{\frac{1}{2}}, \qquad \text{(B 17)}$$

$$\int_{0}^{\infty} j(y,t)\,dy = \tilde{j}_0(0,t) = -1. \qquad \text{(B 18)}$$

The problem now arises: in which regions of the flow will the transforms (2.15) to (2.18) with the value of $\bar{j}_0(k,p)$ given by (B 14) be valid? Evidently the expressions for the vorticity and electric current will be valid for

$$y = O(\beta t)^{\frac{1}{2}}, \quad 0 < 1-|x|/t = O(1)$$

(note $|x|$ may be $O(t)$), since the exponential term dictates that the main contribution to the inversion integrals will be obtained when $p = O(t^{-1})$, $K = O(1)$ (or $K \gg 1$ if $|x| \ll t$, $y = 0$). However, discussion of the velocity distribution is left to § 5.

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